3.2: General Solutions of Linear Equations

Everything that we did in Section 3.1 for second-order linear equations extends in a natural way to n^{th}-order linear equations of the form

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x) \tag{1}$$

or

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x). \tag{2}$$

Again, if $f(x) = 0$ in (2) then the equation is **homogeneous**.

Theorem 1. (Principle of Superposition for Homogeneous Equations)
Let y_1, $y_2, \ldots$, y_n be n solutions to the homogeneous linear equation (2); i.e. $f(x) = 0$. If $c_1, c_2, \ldots, c_n$ are constants, then the linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

is also a solution to (2).

Exercise 1. Verify that $y_1(x) = e^{-3x}$, $y_2(x) = \cos 2x$ and $y_3(x) = \sin 2x$ are all solutions of

$$y^{(3)} + 3y'' + 4y' + 12y = 0.$$

Find the general solution.

$y_1 = e^{-3x}$
$y_1' = -3e^{-3x}$
$y_1'' = 9e^{-3x}$
$y_1^{(3)} = -27e^{-3x}$

$y_2 = \cos 2x$
$y_2' = -2\sin 2x$
$y_2'' = -4\cos 2x$
$y_2''' = 8\sin 2x$

$y_3 = \sin 2x$
$y_3' = 2\cos 2x$
$y_3'' = -4\sin 2x$
$y_3''' = -8\cos 2x$

$y = c_1e^{-3x} + c_2\cos 2x + c_3\sin 2x$

Theorem 2. (Existence and Uniqueness for Linear Equations)
Suppose that $p_1, p_2, \ldots, p_n$ and f are continuous on I containing a. Then, given n numbers $b_1, \ldots, b_{n-1}$, the n^{th}-order linear equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

has a unique solution on I with n initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)} = b)n - 1.$$

Definition 1. The n functions $f_1, \ldots, f_n$ are said to be **linearly independent** on I provided there are no constants $c_1, \ldots, c_n$ (not all zero) such that

$$c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$$

for all $x \in I$.

Example 1. The functions $f_1(x) = \sin 2x$, $f_2(x) = \sin\cos x$, and $f_3(x) = e^x$ are linearly ~~in~~dependent on $\mathbb{R}$ because

$$(1)f_1 + (-2)f_2 + (0)f_3 = 0.$$

Definition 2. Given that $f_1, \ldots, f_n$ are all $(n-1)$ times differentiable, the **Wronskian** is given by

$$W = \det \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}.$$

Theorem 3. (Wronskian of Solutions)
Suppose that $y_1, \ldots, y_n$ are n solutions to the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

on an open interval I, where each p_i is continuous.

(a) If $y_1, \ldots, y_n$ are linearly dependent, then $W \equiv 0$ on I.

(b) If $y_1, \ldots, y_n$ are linearly independent, then $W \neq 0$ at each $x \in I$.

Exercise 2. Use Theorem 3 to verify the linear independence and linear dependence of Exercise 1 and Example 1 respectively.

$$\det\begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix} = e^{-3x}(8\sin^2 2x + 8\cos^2 2x)$$

$$- \cos 2x\,(12e^{-3x}\sin 2x - 18e^{-3x}\cos 2x)$$

$$+ \sin 2x\,(12e^{-3x}\cos 2x - 18e^{-3x}\sin 2x)$$

$$= e^{-3x}(8 + 18(\cos^2 2x - \sin^2 x)) \neq 0 \text{ anywhere}$$

$$\det\begin{vmatrix} \sin 2x & \sin x\cos x & e^x \\ 2\cos 2x & \cos^2 x - \sin^2 x & e^x \\ -4\sin 2x & -4\cos x\sin x & e^x \end{vmatrix} = \sin 2x\,(e^x(\cos^2 x - \sin^2 x) + e^x 4\cos x\sin x)$$

$$- \sin x\cos x\,(2e^x\cos 2x + 4e^x\sin 2x)$$

$$+ e^x(-8\cos 2x\cos x\sin x + 4\sin 2x(\cos^2 x - \sin^2 x))$$

$$= 0 \quad \text{(from trig identities)}$$

Theorem 4. (General Solutions of Homogeneous Equations)
Let $y_1, \ldots, y_n$ be n linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

If y is any solution to this equation, then there exists constants $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

for all $x \in I$.

Consider the general n^{th}-order linear equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x).$$

Call the a solution to this equation y_p, the **particular solution.** If we were to add any solution of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

to y_p we would obtain another solution to the original equation. The solutions to the homogeneous equation are therefore called **complimentary solutions** and are often denoted by y_c. Notice that the general form of y_c is given by Theorem 4.

Theorem 5. (Solutions of Nonhomogeneous Equations)
Let y_p be a particular solution of the nonhomogeneous equation (2) on the interval I, where each p_i and f are continuous. Let $y_1, \ldots, y_n$ be n linearly independent solutions of the associated homogeneous equation. Then for any solution y, there exists constants $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x) = y_c(x) + y_p(x)$$

for all $x \in I$.

Exercise 3. It is evident that $y_p(x) = 3x$ is a particular solution of the equation

$$y'' + 4y = 12x,$$

and that $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$ is its complimentary solution. Find a solution that satisfies the initial conditions $y(0) = 5$, $y'(0) = 7$.

$$y(x) = 3x + c_1 \cos 2x + c_2 \sin 2x$$

$$y(0) = 5 = c_1$$

$$y'(x) = 3 - 2c_1 \sin 2x + 2c_2 \cos 2x$$

$$= 3 - 10 \sin 2x + 2c_2 \cos 2x.$$

$$y'(0) = 7 = 3 - 2c_2 \Rightarrow c_2 = -2.$$

So

$$y(x) = 3x + 5\cos 2x - 2\sin 2x.$$

Homework. 1, 7-17, 21-25 (odd)